Recent Advances in Lift-and-Project

by

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Abstract

In recent years the lift-and-project approach has been used successfully within a branch-and-cut framework to solve large, difficult pure and mixed 0-1 programs that have resisted solution efforts by pure branch and bound codes. The approach uses a linear description in a higher dimensional space of the convex hull of the disjunctive set created by imposing one or several 0-1 conditions. By solving a linear program derived from this higher dimensional representation – the cut generating linear program (CGLP) – the standard lift-and-project procedure obtains a deepest cut in a well defined sense. We propose a modification of CGLP that allows us to generate not just one deepest cut, but a class of cuts with desirable properties, each at the cost of one extra pivot in the optimal tableau of the modified CGLP.

This paper investigates a certain enhancement of the lift-and-project approach. The underlying research has its immediate background and starting point in the work reported in [5] and [6], and is part of a larger project pursued jointly with Sebastian Ceria and Gerard Cornuéjols. It has some connections with the work of Lovász and Schrijver [11] on matrix cones, the work of Sherali and Adams [12], but primarily with the work on disjunctive programming [1, 2, 3] done in the 70's. An early version of the material discussed here was circulated under [4]. For some related work on lift and project see [7, 9, 10, 13].

The lift-and-project approach to mixed 0-1 programming relies primarily on the following two ideas (results), both of which have their roots in the work on disjunctive programming, i.e. on optimization over unions of polyhedra, done in the seventies ([1]; see also [3] for 1 below, and [2] for 2 below).

1. There is a compact representation of the convex hull of a union of polyhedra. Namely, given polyhedra $P_i := \{x \in I\!R^n : A^i x \geq b^i\} \neq \emptyset$, $i \in Q$, the closed convex hull of $\cup_{i \in Q} P_i$ is the set of those $x \in I\!R^n$ for which there exists vectors $(y^i, y_0^i) \in I\!R^{n+1}$, $i \in Q$ such that

$$\begin{array}{rcll} x - \sum(y^i : i \in Q) & = & 0 & \\ A^i y^i - b^i y_0^i & \geq & 0 & i \in Q \\ y_0^i & \geq & 0 & \\ \sum(y_0^i : i \in Q) & = & 1. & \end{array}$$

   The number of variables and constraints in this representation is linear in the number $|Q|$ of polyhedra in the union.

2. The closed convex hull of the feasible 0-1 points of a mixed 0-1 program

$$\min\{cx : Ax \geq b,\ x \geq 0,\ x_j \in \{0,1\},\ j \in N_1 \subseteq N\} \tag{1}$$

can be generated by imposing the 0-1 conditions successively. Namely, let $S_1, \ldots, S_p$ be an arbitrary partition of N_1, and for $i = 1, \ldots, p$, let Q_i index the collection of 0-1 vectors with components in S_i (i.e., $|Q_i| = 2^{|S_i|}$). Further, let

$$K_0 := \{x \in I\!R^n : Ax \geq b,\ x \geq 0,\ x_j \leq 1,\ j \in N_1\}$$

and define recursively

$$K_i := conv(K_{i-1} \cap \{x \in I\!R^n : x_j \in \{0, 1\},\ j \in S_i\},$$

$i = 1, \ldots, p$, where for any set W, $conv(W)$ denotes the closed convex hull of W. Then

$$K_p = conv(K_0 \cap \{x \in I\!R^n : x_j \in \{0, 1\},\ j \in N_1\}).$$

Conceptually, the lift-and-project approach is supposed to work as follows. Let $\tilde{A}x \geq \tilde{b}$ denote the system in $I\!R^n$ defining K_0.

0.1. Solve the LP over K_0

0.2. Choose $S_1 \subset N_1$ and *lift* the problem into $I\!R^{n+|Q_1|(n+1)}$ by replacing $\tilde{A}x \geq \tilde{b}$ with a linear system in x and $\{y^i, y_0^i\}_{i \in Q_1}$ defining the convex hull of $K_0 \cap \{x : x_j \in \{0, 1\},\ j \in S_1\}$.

0.3. *Project* the system onto $I\!R^n$ (the x-space). Let $\alpha^h x \geq \beta^h$, $h \in T_1$, be the set of projected inequalities that together with $\tilde{A}x \geq \tilde{b}$ define K_1.

Iterate the procedure. At step t, do

t.1. Solve the LP over K_{t-1}.

t.2. Choose $S_t \subset N_1 \setminus \cup_{i=1}^{t-1} S_i$ and lift the problem into $I\!R^{n+|Q_t|(n+1)}$ by replacing the constraints of K_{t-1} with the linear system in x and $\{y^i, y_0^i\}_{i \in Q_t}$ defining the closed convex hull of $K_{t-1} \cap \{x : x_j \in \{0, 1\},\ j \in S_t\}$.

t.3. Project the system onto $\mathbb{R}^n$ (the x-space). Let $\alpha^h x \geq \beta^h$, $h \in T_t$, be the set of projected inequalities that together with those of K_{t-1} define K_t.

After p iterations, this procedure should in principle yield the optimum over the closed convex hull of K_p, i.e. the solution to the mixed integer program.

In practice, such an approach is not workable, since at every iteration the number of projected inequalities $\alpha^h x \geq \beta^h$, $h \in T_t$, is exponential in n. Therefore the standard lift-and-project procedure of [5], [6] generates just one member of the family indexed by T_t, namely the inequality that provides a "deepest cut" in that it cuts off the optimal LP solution $\bar{x}$ over K_{t-1} by a maximum amount.

This procedure, suitably embedded into a branch-and-cut framework where cuts are generated at various nodes of the search tree and made globally valid through a cut-lifting step, has been implemented and extensively tested; primarily in [6], but also in [7, 8, 14] among others. In [6], the branch-and-cut code MIPO using lift-and-project cuts was tested on a battery of 29, mostly difficult, mixed 0-1 programs obtained from MIPLIB and from the literature, and shown to be able to solve all 29 problems in computing times that compare favorably with OSL, CPLEX and MINTO. In [7] the same approach with some added features was tested on maximum clique problems from the Second DIMACS Challenge. In [14], S. Thienel compares the performance of his ABACUS branch-and-cut code in two different modes of operation, one using lift-and-project cuts and the other using Gomory cuts, with the outcome that the version using lift-and-project cuts is considerably faster on all hard problems, where hard means requiring 10 or more minutes. Finally, in [8] the authors report that in the framework of a parallel branch and cut code which uses a variety of cuts, adding lift-and-project cuts for the harder problems helped solve them faster.

As mentioned earlier, the standard lift-and-project approach generates one deepest cut whenever it solves a cut-generating linear program. The objective of the research discussed

here is to find a middle ground between the two extremes of generating all the inequalities $\alpha^h x \geq \beta^h$, $h \in T_t$, and generating just one deepest cut. This middle ground is defined as generating all those inequalities of the family indexed by T_t that are tight for the optimal solution $\bar{x}$ of K_t.

Consider the mixed 0-1 program (1) and its linear programming relaxation

$$\begin{aligned} &\min\{cx : Ax \geq b,\ x_j \leq 1,\ j \in N_1,\ x_j \geq 0,\ j \in N\} \\ &= \min\{cx : \tilde{A}x \geq \tilde{b}\}. \end{aligned} \tag{LP}$$

Let $S \subseteq N_1$ and define

$$P_S := conv\ \{x \in I\!R^n : \tilde{A}x \geq \tilde{b},\ x_j \in \{0,1\} \text{ for } j \in S\}.$$

Then there exist inequalities $\alpha^h x \geq \beta^h$, $h \in T$, such that

$$P_S := \{x \in I\!R^n : \ \alpha^h x \geq \beta^h,\ h \in T\},$$

but these inequalities are exponentially many and not readily available.

Problem. Given $\bar{x}$ such that

$$c\bar{x} = \min\ \{cx : x \in P_S\},$$

find the inequalities $\alpha^h x \geq \beta^h$, $h \in T' \subset T$ that are tight for $\bar{x}$.

Application. If S is "small", $\bar{x}$ can be found, for instance, by branch and bound. Solving the above Problem would then enable us to replace (LP) by min $\{cx : \tilde{A}x \geq \tilde{b},\ \alpha^h x \geq \beta^h,\ h \in T'\}$, whose solution $\bar{x}$ has $\bar{x}_j \in \{0,1\}$, $j \in S$, and repeat the procedure for some $S'' \subset N_1 \setminus S$. In other words, solving the above Problem is a way of "consolidating" the results of a partial branch and bound run into a tighter LP relaxation.

Discussion. Let Q index the collection of all 0-1 vectors with components in S, i.e. $|Q| =$

$2^{|S|}$. Consider the extended formulation of P_S in $n+|Q|(n+1)$-dimensional space:

$$\begin{array}{rclcccl} P_S & = & \{x \in I\!R^n : x & - & \sum_{i\in Q} y^i & & = 0 \\ & & & & \tilde{A}y^i & - \quad \tilde{b}^i y_0^i & \geq 0, \quad i \in Q \\ & & & & & \sum_{i\in Q} y_0^i & = 1 \qquad \}. \end{array}$$

Here $\tilde{b}^i := \binom{b}{\delta^i}$, where δ^i is defined as follows. If the original right hand side is written as $\tilde{b} := \binom{b}{\delta^0}$, where $\delta^0 \in \{0,-1\}^{n+p}$, with the first n components equal to 0 and the remaining p components equal to -1, then δ^i corresponding to the i-th 0-1 vector indexed by Q is obtained from δ^0 by changing the -1 in δ^0 to 0 in δ^i for all those variables that have to be forced to 0, and changing the 0 in δ^0 to 1 in δ^i for all those variables to be forced to 1. Notice that the constraints $y_0^i \geq 0$, $i \in Q$, have been omitted, as they are implied by the other inequalities.

To express P_S in x-space, consider the projection cone

$$W := \left\{ (\alpha, \{u^i\}_{i\in Q}, \beta) \;\middle|\; \begin{array}{lclcll} \alpha & - & u^i\tilde{A} & & = 0 & \\ & & u^i\tilde{b}^i & - \quad \beta & = 0 & i \in Q \\ & & u^i \geq 0 & & & \end{array} \right\}.$$

The inequalities $\alpha^h x \geq \beta^h$ defining P_S in x-space correspond to the (α,β)-components of the extreme rays of W. The task is to identify those extreme rays of W that give rise to inequalities $\alpha^h x \geq \beta^h$ satisfied at equality by $\bar{x}$.

Solution. Define the linear program

$$\begin{array}{rcccl} \max \beta & - & \alpha\bar{x} & & \\ & - & \alpha & + \quad u^i\tilde{A} & = 0 \\ \beta & & & - \quad u^i\tilde{b}^i & = 0 \quad i \in Q \\ & & & \sum_{i\in Q} u^i e & \leq k \\ & & u^i \geq 0, & i \in Q & \end{array} \qquad (\mathrm{LD1}(\bar{x}))$$

where $k > 0$ and $e = (1,\ldots,1)$.

Theorem 1 *The inequalities $\alpha^h x \geq \beta^h$ of P_S satisfied at equality by $\bar{x}$ are precisely those corresponding to the (α, β)-components of optimal solutions to (LD1($\bar{x}$)) such that $\alpha \neq 0$.*

Proof. The feasible set of (LD1($\bar{x}$)) is the cone W truncated by the inequality $\sum_{i \in Q} u_i e \leq k$, a normalization device. Basic solutions to (LD1($\bar{x}$)) such that $\alpha \neq 0$ are in 1-1 correspondence with extreme rays of W. Consider now the linear program dual to (LD1($\bar{x}$)),

$$\begin{array}{rcccccll}
 & & \min k y_0^0 & & & & & \\
-\sum_{i \in Q} y^i & & & & = & -\bar{x} & & \\
\tilde{A} y^i & - & \tilde{b}^i y_0^i & + \; e y_0^0 & \geq & 0, & i \in Q & \quad (\text{LP1}(\bar{x})) \\
 & & \sum_{i \in Q} y_0^i & & = & 1 & & \\
 & & & y_0^0 & \geq & 0. & &
\end{array}$$

(LP1($\bar{x}$)) has an optimal solution of the form $\bar{y}^{i_*} = \bar{x}$, $\bar{y}_0^{i_*} = 1$ for the particular $i_* \in Q$ that corresponds to $\bar{x}$ (i.e. for the unique i_* such that $D^{i_*}\bar{x} \geq d^{i_*}$), and $(\bar{y}^i, \bar{y}_0^i) = (0, 0)$ for all $i \in Q \setminus \{i_*\}$, $\bar{y}_0^0 = 0$. Hence (LP1($\bar{x}$)) has an optimal solution with value 0. Further, notice that y_0^0 can be pivoted into the basis with value 0, i.e. without changing the solution. It follows that (LD1($\bar{x}$)) has an optimal solution of value 0 for with $\sum_{i \in Q} u^i e = k$, i.e. (LD1($\bar{x}$)) has a nonzero optimal solution with value 0. The (α, β)-component of such a solution defines an inequality satisfied at equality by $\bar{x}$ if and only if $\alpha \neq 0$.□

Thus one way of solving our problem is to generate nonzero basic optimal solutions to (LD1($\bar{x}$)) satisfying $\sum_{i \in Q} u^i e = k$. Although (LD1($\bar{x}$)) is large, it has a strong structure that can be exploited.

Working in a subspace. We should mention at this point that, as in the case of the lift-and-project procedure described in [5], (LD1($\bar{x}$)) can be restricted to the relevant subspace, and the inequalities obtained in this fashion can be lifted into the full space by the same technique as in [5]. For our purposes F, the index set of the relevant subspace, will be defined slightly differently, so as to always contain S, although the components of $\bar{x}$ indexed by S

are not fractional. Also, we will assume w.l.o.g. that for $j \in S$, $\tilde{x}_j = 1$ implies that $\tilde{x}_j$ is basic; and for $j \notin S$, if $\tilde{x}_j$ is nonbasic then $\tilde{x}_j = 0$ (i.e. all nonbasic $\tilde{x}_j$ at their upper bound, if any, have been complemented). We define

$$F := S \cup \{j \in N \setminus S : \tilde{x}_j > 0\},$$

and we will denote by $\tilde{x}_F$ the subvector of $\tilde{x}$ with components indexed by F. The matrix $\tilde{A}_F$ is obtained from the matrix $\tilde{A}$ by deleting the columns indexed by $N \setminus F$ and the rows corresponding to the inequalities $x_j \geq 0$ and $-x_j \geq -1$ for $j \in N \setminus F$. Further, the vectors $\tilde{b}_F$ and $\tilde{b}^i_F$, $i \in Q$, are obtained from $\tilde{b}$ and $\tilde{b}^i$, $i \in Q$, by removing the components corresponding to the inequalities $x_j \geq 0$ and $-x_j \geq -1$ for $j \in N \setminus F$. The problem (LD1($\tilde{x}$)) can then be replaced by

$$\begin{array}{rrrrrrl}
max\ \beta & - & \alpha\tilde{x}_F & & & & \\
 & - & \alpha & + & u^i\tilde{A}_F & = & 0 \\
\beta & & & - & u^i\tilde{b}^i_F & = & 0 \qquad i \in Q \\
 & & & & \sum_{i \in Q} u^i e & \leq & k \\
 & & u^i \geq 0, & & i \in Q & &
\end{array} \qquad \text{(LD1}(\tilde{\mathrm{x}}_{\mathrm{F}}))$$

where α and u^i, $i \in Q$, have been redimensioned according to $\tilde{A}_F$ and are to be read as shorthand for α_F and u^i_F, $i \in Q$.

Elimination of (α, β)**.** Now let $i_* \in Q$ be the index associated with $\tilde{x}$, i.e. the (unique) i such that $\tilde{A}y^i \geq \tilde{b}^i$ is satisfied by $\tilde{x}$. Using $\alpha = u^{i_*}\tilde{A}_F$ and $\beta = u^{i_*}\tilde{b}^{i_*}_F$ to eliminate α and β, and imposing equality in the last inequality, (LD1($\tilde{x}_F$)) can be restated as

$$\begin{array}{lrrrl}
\min u^{i_*}(\tilde{A}_F\tilde{x}_F - \tilde{b}^{i_*}_F) & & & & \\
u^{i_*}\tilde{A}_F & - & u^i\tilde{A}_F & = & 0 \\
-\, u^{i_*}\tilde{b}^{i_*}_F & + & u^i\tilde{b}^i_F & = & 0 \qquad i \in Q \setminus \{i_*\} \\
 & & \sum_{i \in Q} u^i e & = & k \\
 & & u^i \geq 0, i \in Q & &
\end{array} \qquad \overline{\text{(LD1}(\tilde{\mathrm{x}}_{\mathrm{F}}))}$$

Proposition 2 *A feasible solution* $\{u^i\}_{i \in Q}$ *to* $\overline{(\text{LD1}(\tilde{x}_F))}$ *is optimal if and only if*

$$(\tilde{A}_F\tilde{x}_F)_j > (\tilde{b}^{i_*}_F)_j \Rightarrow u^{i_*}_j = 0.$$

Proof. From the foregoing discussion, the optimum of $\overline{(\mathrm{LD1}(\tilde{x}_F))}$ has value 0.□

While $\overline{(\mathrm{LD1}(\tilde{x}_F))}$ has fewer variables than $(\mathrm{LD1}(\tilde{x}_F))$, this latter formulation has the advantage of having the values of α and β readily available when needed. Further, since α, β are unconstrained in sign, they can be made part of the starting basis when solving $(\mathrm{LD1}(\tilde{x}_F))$ and kept in the basis till the end, so that their presence does not affect the number of pivots. Therefore we will prefer to work with $(\mathrm{LD1}(\tilde{x}_F))$.

Generating inequalities. It is easy to see that optimal solutions to $(\mathrm{LD1}(\tilde{x}_F))$ are highly dual degenerate, i.e. there are typically many different optimal solutions. In fact, from the proof of the above Theorem, it follows that all the reduced costs corresponding to u_j^i for all j and $i \in Q \setminus \{i_*\}$ are 0, since these reduced costs are the slack variables of the inequalities $\tilde{A}^i y^i - \tilde{b}^i y_0^i \geq 0$ of $(\mathrm{LP1}(\tilde{x})_F)$.

It should be noted at this point that while the complete set of inequalities $\alpha^h x \geq \beta^h$, $h \in T'$, tight for $\tilde{x}$, obviously cuts off the optimal solution $\tilde{x}$ to (LP), this is not necessarily the case for each individual inequality indexed by T'. In order to generate inequalities that are guaranteed to cut off $\tilde{x}$, one may add to the constraints of $(\mathrm{LD1}(\tilde{x}_F))$ the inequality $\beta - \alpha\tilde{x}_F \geq \epsilon$ with some small $\epsilon > 0$. One advantage of doing this is that it also guarantees $\alpha \neq 0$. Thus generating the inequalities $\alpha x \geq \beta$ of P_S that are tight for $\tilde{x}$ can be done by the following procedure:

1. Solve $(\mathrm{LD1}(\tilde{x}_F))$.

2. Find alternate optimal solutions by pivoting into the basis, one at a time, nonbasic variables with zero reduced cost for which the minimum ratio (of the primal simplex method) is nonzero.

3. For each new vector (α, β) obtained in this way, use the "minimum required angle" criterion to discard inequalities too close to some earlier inequality.

4. Lift each inequality into the full space.

An efficient implementation of this procedure may take advantage of the special structure of LD1($\tilde{x}_F$). Although this linear program is relatively large ($|Q|+1$ times the size of the original LP restricted to the subspace of the variables indexed by F), on the other hand is highly structured. Indeed, the coefficient matrix of LD1($\tilde{x}_F$) has the general form

$$
\begin{array}{c}
\hspace{4em}\overbrace{\hspace{12em}}^{|Q| \text{ blocks}} \\
\left(\begin{array}{cc|cccc}
0 & -I & \tilde{A}_F^T & 0 & \cdots & 0 \\
0 & -I & 0 & \tilde{A}_F^T & \cdots & 0 \\
\vdots & \vdots & & & \ddots & \\
0 & -I & 0 & 0 & \cdots & \tilde{A}_F^T \\
\hline
1 & 0 & -\tilde{b}^{1T} & 0 & & 0 \\
1 & 0 & 0 & -\tilde{b}^{2T} & & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & \cdots & -\tilde{b}^{|Q|T}
\end{array}\right)
\end{array}
$$

where the $|Q|$ blocks in the upper part of the matrix are all identical, and the differences are restricted to the last $|Q|$ rows of the matrix, containing the right hand side of each term in the disjunction.

Note that in the absence of any special structure $|Q| = 2^{|S|}$, i.e. the number Q of terms in the disjunction underlying LD1($\tilde{x}_F$) is exponential in the number $|S|$ of 0-1 variables being arbitrated. However, the presence of some structure can cut down this number drastically, in some cases making $|Q|$ linear, rather than exponential, in $|S|$. For instance, if the system $Ax \geq b$ contains an inequality of the form $x(S) \leq 1$, i.e. a generalized upper bounding constraint, then $|Q| = |S| + 1$, since x_S, the subvector of x with components indexed by S, is either one of the $|S|$-dimensional unit vectors or the $|S|$-dimensional zero vector.

Example 1.

$$\begin{array}{lrrrrrrrrr}
\min & 2x_1 & + & x_2 & + & 3x_3 & + & x_4 & & \\
\text{s.t.} & & & & & & & & & \\
 & 4x_1 & + & 4x_2 & + & 3x_3 & - & x_4 & \geq & 3 \\
 & -x_1 & + & 2x_2 & - & 2x_3 & + & 3x_4 & \geq & 1 \\
 & x_1 & - & x_2 & + & 2x_3 & + & 2x_4 & \geq & 1 \\
 & \multicolumn{9}{l}{x \geq 0, \quad x_j \in \{0, 1\}, \ j = 1, 2, 3}
\end{array}$$

The optimal solution to the LP relaxation (in which $x_j \in \{0, 1\}$ is replaced by $0 \leq x_j \leq 1$, $j = 1, 2, 3$) is $\bar{x} = (0.578,\ 0.270,\ 0,\ 0.346)$.

We choose $S := \{1, 2\}$, i.e. impose the condition $x_j \in \{0, 1\}$ for $j = 1, 2$, and we find the optimal solution $\tilde{x} = (0, 1, 0, 1)$ to the LP relaxation subject to this condition. Thus we define

$$\begin{aligned}
F &:= S \cup \{j \in N \setminus S : \tilde{x}_j > 0\} \\
&= \{1, 2, 4\},
\end{aligned}$$

and work with the problem (LD1($\tilde{x}_F$)):

$$\begin{array}{lllllllll}
\min \beta & - \alpha\tilde{x} & & & & & & & \\
 & -\alpha & + \tilde{A}_F^T u^1 & & & & & = & 0 \\
 & -\alpha & & + \tilde{A}_F^T u^2 & & & & = & 0 \\
 & -\alpha & & & + \tilde{A}_F^T u^3 & & & = & 0 \\
 & -\alpha & & & & + \tilde{A}_F^T u^4 & & = & 0 \\
\beta & & - \tilde{b}_F^{1T} u^1 & & & & & = & 0 \\
\beta & & & - \tilde{b}_F^{2T} u^2 & & & & = & 0 \\
\beta & & & & - \tilde{b}_F^{3T} u^3 & & & = & 0 \\
\beta & & & & & - \tilde{b}_F^{4T} u^4 & & = & 0 \\
 & & e^T u^1 & + e^T u^2 & + e^T u^3 & + e^T u^4 & & = & 10 \\
 & & \multicolumn{4}{l}{u^1, u^2, u^3, u^4 \geq 0} & & &
\end{array}$$

Here

$$\tilde{x}_F^T = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_4)^T = (0, 1, 1)^T,$$

and

$$\tilde{A}_F = \left|\begin{array}{ccc} 4 & 4 & -1 \\ -1 & 2 & 3 \\ 1 & -1 & 2 \\ 1 & & \\ & 1 & \\ & & 1 \\ -1 & & \\ & -1 & \end{array}\right| \quad b_F^1 = \left|\begin{array}{c} 3 \\ 1 \\ 1 \\ \\ \\ \\ \\ \\ \end{array}\right| \quad b_F^2 = \left|\begin{array}{c} 3 \\ 1 \\ 1 \\ 1 \\ \\ \\ -1 \\ \\ \end{array}\right| \quad b_F^3 = \left|\begin{array}{c} 3 \\ 1 \\ 1 \\ \\ 1 \\ \\ \\ -1 \end{array}\right| \quad b_F^4 = \left|\begin{array}{c} 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ \\ -1 \\ -1 \end{array}\right|$$

where the blanks represent zeros.

Solving (LD1($\tilde{x}_F$)) yields an optimal solution whose (α, β) components are

$$\alpha_1 = 0.559, \ \alpha_2 = 0.102, \ \alpha_4 = 1.119, \ \beta = 1.221.$$

The lifting coefficient for α_3 is easily seen to be 0, and dividing through with β to obtain a right hand side of 1 yields the cut

$$0.458x_1 + 0.084x_2 + 0.916x_4 \geq 1.$$

Performing several pivots in the optimal tableau according to the rules discussed above yields the additional optimal solutions

$\underline{\alpha_1}$	$\underline{\alpha_2}$	$\underline{\alpha_4}$	$\underline{\beta}$
0.875	0.448	0.957	1.407
0.654	0.218	1.307	1.525
0.976	0.650	0.976	1.626

In each case, the lifting coefficient for α_3 is 0, and dividing through with β we obtain the following three additional cuts:

$$\begin{array}{ccccccc} 0.623x_1 & + & 0.319x_2 & + & 0.681x_4 & \geq & 1 \\ 0.429x_1 & + & 0.143x_2 & + & 0.857x_4 & \geq & 1 \\ 0.600x_1 & + & 0.400x_2 & + & 0.600x_4 & \geq & 1 \end{array}$$

Each of the above four cuts is satisfied at equality by $\tilde{x}$. Each of them happens to cut off $\tilde{x}$.$\square$

Duplication of inequalities. One difficulty with the procedure outlined above is that it will frequently generate multiples of the same inequality. This can happen in two different ways:

1. A pivot may change some or all of the variables u^i, $i \in Q$, but leave the components of α, β unchanged.

2. A pivot may change the components of (α, β) but change them proportionally, i.e. it may leave the ratios α_j/β, $j \in N$, unchanged.

Both kinds of difficulties can be handled by an appropriate check of the pivot column, say g. To avoid problem 1, one has to make sure that g has a nonzero entry in at least one of the rows associated with the components of (α, β). To avoid problem 2, one has to make sure that the ratios $\alpha_1/g_{\alpha_1}, \ldots, \alpha_n/g_{\alpha_n}, \beta/g_\beta$ are not all equal. Here g_{α_i} (g_β) denotes the entry of g in the row containing α_i (β).

It is not always possible to avoid a pivot which changes the u^i but leaves the cut $\alpha x \geq \beta$ essentially unchanged. If necessary, such a pivot must be performed in order to obtain another working basis that yields access to a new set of pivots.

Generating facets of P_S. Under what circumstances do the inequalities generated by the above procedure define facets of P_S?

To answer this question, we note first that when P_S is full dimensional, then the reverse polar cone

$$P_S^\sharp := \{(\alpha, \beta) \in \mathbb{R}^{n+1} : \alpha x \geq \beta \text{ for all } x \in P_S\}$$

of P_S is pointed; which means that $P_S^\sharp$ is the conical hull of its extreme rays. Assume for the time being that this is the case. Then the facets of P_S are in a one to one correspondence with the extreme rays of $P_S^\sharp$.

On the other hand, from the representation of P_S in $I\!R^{n+|Q|(n+1)}$ given at the beginning of our discussion, it follows that $P_S^\sharp$ also has a higher dimensional representation, namely

$$P_S^\sharp = \left\{ (\alpha,\beta) \in I\!R^{n+1} \;\middle|\; \begin{array}{lcccl} \exists u^i \geq 0, \; i \in Q, & & \text{such that} & & \\ \alpha & - & u^i \tilde{A} & = & 0 \\ & \beta \; - & u^i \tilde{b}^i & = & 0, \;\; i \in Q \end{array} \right\}.$$

The constraint set defining this higher dimensional representation differs from that of LD1($\tilde{x}$) only in the absence of the normalization inequality. Adding this inequality replaces the cone $P_S^\sharp$ with a polyhedron

$$\tilde{P}_S^\sharp := \left\{ (\alpha,\beta) \in I\!R^{n+1} \;\middle|\; \begin{array}{lcccl} \alpha & - & u^i \tilde{A} & = & 0 \\ & \beta \; - & u^i \tilde{\beta} & = & 0 \;\;\; i \in Q \\ & \sum_{i\in Q} & u^i e & \leq & k \\ & & u^i \geq 0, & & i \in Q \end{array} \right\}$$

whose extreme points (vertices) are in one to one correspondence with the extreme rays of the cone $P_S^\sharp$, hence with the facets of P_S.

But the polyhedron $\tilde{P}_S^\sharp$ is nothing but the projection onto the subspace of the (α,β) variables of the feasible set of (LD1($\tilde{x}$)). Thus the question that we started out with reduces to the following: when does an extreme point (basic solution) $(\alpha^0, \beta^0, \{u^{i0}\}, \; i \in Q)$ of the feasible set of (LD1($\tilde{x}$)) project into an extreme point (α^0,β^0) of $\tilde{P}_S^\sharp$?

One answer to this question is that the (α^0,β^0)-component of a basic feasible solution to (LD1($\tilde{x}$)) is an extreme point of $\tilde{P}_S^\sharp$ if and only if there exists a linear function of (α,β) that attains its unique maximum at $(\alpha,\beta) = (\alpha^0,\beta^0)$. A basic feasible solution to (LD1($\tilde{x}$)) that satisfies this criterion will be called *regular* (see [1] for a discussion). To check the regularity of a basic feasible solution to LD1($\tilde{x}$)), one may want to carry a second objective function row, in which the vector $(1,\tilde{x})$ is replaced by one, if it exists, that makes the (α,β) component of an optimal solution unique, i.e. produces nonzero reduced costs for all nonbasic variables whose pivoting into the basis would affect the (α,β)-component. Methods for finding such a

vector or showing that none exists are currently under investigation. So far we have assumed that P_S is full dimensional. When this is not the case, i.e. $\dim P_S = d < n$, then $P_S^\sharp$ is not pointed, hence has no extreme rays; and the dimension of its lineality space L is $n-d$. In this case there is a one to one correspondence between facets of P_S and $(n-d+1)$-dimensional faces of $P_S^\sharp$. Since the latter are not easy to handle, it is preferable in this case to work with $P_S^\sharp \cap L^\perp$, where $L^\perp$ is the orthogonal complement of the lineality space L of $P_S^\sharp$. $P_S^\sharp \cap L^\perp$ is a pointed cone whose extreme rays are in a one to one correspondence with the $(n-d+1)$-dimensional faces of $P_S^\sharp$, hence with the facets of P_S. The analytical expression for this cone is

$$P_S^\sharp \cap L^\perp = \left\{ (\alpha,\beta) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{rcl} \alpha x & \geq & \beta \quad \text{for all } x \in P_S \\ \alpha v & = & 0 \quad \text{for all } v \in L \end{array} \right\},$$

where "all $v \in L$" can be replaced by "all $v \in B(L)$" for some basis $B(L)$ of L. Note that $\dim L = n-d$ and thus a basis $B(L)$ has $n-d$ elements.

When it comes to the polyhedron $\tilde{P}_S^\sharp$, the normalization inequality $\sum_{i\in Q} u^i e \leq k$ effectively bounds all the variables, including the components of (α,β); and thus the smallest dimensional, i.e. $(n-d+1)$-dimensional, faces of $P_S^\sharp$ correspond to extreme points of $\tilde{P}_S^\sharp$. However, this correspondence is no longer one to one, as the same facet of P_S can now be defined by different inequalities $\alpha x \geq \beta$, corresponding to different extreme points of $\tilde{P}_S^\sharp$. Nevertheless, among all the equivalent inequalities $\alpha x \geq \beta$ defining the same facet of P_S, there is only one (modulo a multiplicative factor) whose normal lies in $L^\perp$. If we amend the constraint set of $\tilde{P}_S^\sharp$ by the system of equations

$$\alpha v = 0 \text{ for all } v \in B(L),$$

i.e. replace $\tilde{P}_S^\sharp$ with $\tilde{P}_S^\sharp \cap L^\perp$, we make sure that the correspondence between extreme points of $P_S^\sharp \cap L^\perp$ and facets of P_S are one to one.

We illustrate this last point on an example.

Example 2. The traveling salesman polytope P defined on the complete directed graph G

with 4 nodes has dimension 5 (= number of variables (12) minus rank of equality system (2×4 - 1)). The 6 (affinely independent) tours of G are $(1,2,3,4)$, $(1,2,4,3)$, $(1,3,2,4)$, $(1,3,4,2)$, $(1,4,2,3)$, $(1,4,3,2)$. One of the facets of P is defined by the inequality

$$x_{12} + x_{13} + x_{32} + x_{34} + x_{43} \leq 2,$$

known as T_2, which is satisfied at equality by 5 of the 6 tours, the exception being $(1,4,2,3)$. However, the same facet is also defined by each of the lifted 3-cycle inequalities

$$\begin{array}{ccccccccc} x_{12} & + & x_{24} & + & x_{41} & + & 2x_{21} & \leq & 2, \\ x_{13} & + & x_{34} & + & x_{41} & + & 2x_{43} & \leq & 2, \\ x_{24} & + & x_{43} & + & x_{32} & + & 2x_{34} & \leq & 2, \end{array}$$

which are easily seen to be satisfied at equality by exactly the same 5 tours as the T_2 inequality. However, none of the above four inequalities has its normal vector in the subspace $L^\perp$ generated by the tours. Indeed, every vector in $L^\perp$ satisfies all the equations of the form

$$\begin{array}{rcl} \text{outdegree of } i & = & \text{outdegree of } j \\ \text{indegree of } i & = & \text{indegree of } j \\ \text{outdegree of } i & = & \text{indegree of } j \\ \text{indegree of } i & = & \text{outdegree of } j, \end{array}$$

just as every tour does: the normals of these equations are all in L, the lineality space of $P^\sharp$. However, none of the four inequalities defining our facet satisfies this system of equations. The unique inequality that defines the same facet satisfies the system is

$$3x_{12} + 2x_{13} + 3x_{21} + 2x_{24} + 2x_{32} + 3x_{34} + 2x_{41} + 3x_{43} \leq 8.$$

Indeed, it is easy to check that this inequality (i) is valid, (ii) is satisfied at equality by the same 5 tours as the above four inequalities, and (iii) satisfies the system of equations defining $L^\perp$.□

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In recent years the lift-and-project approach has been used successfully within a branch-and-cut framework to solve large, difficult pure and mixed 0-1 programs that have resisted solution efforts by pure branch and bound codes. The approach uses a linear description in a higher dimensional space of the convex hull of the disjunctive set created by imposing on or several 0-1 conditions. By solving a linear program derived from this higher dimensional representation - the cut generating linear program (CGLP) - the standard lift-and-project procedure obtains a deepest cut in a well defined sense. We propose

a modification of CGLP that allows us to generate not just one deepest cut, but a class of cuts with desirable properties, each at the cost of one extra pivot in the optimal tableau of the modified CGLP.